

# FILLING OF CLOSED SURFACES

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**ABSTRACT.** Let  $F_g$  denote the closed oriented surface of genus  $g$ . A set of simple closed curves is called a filling of  $F_g$  if the complement is the disjoint union of discs. The mapping class group  $\text{Mod}(F_g)$  of genus  $g$  acts on the set of fillings of  $F_g$ . The union of the curves in a filling form a graph on the surface which is so called decorated fat graph. We show that two fillings of  $F_g$  are in the same  $\text{Mod}(F_g)$ -orbit if and only if the corresponding fat graphs are isomorphic. Next, we prove that any filling of  $F_2$  with complement a single disc (i.e., a so called minimal filling) has either three or four closed curves. Moreover, we prove that there exist unique  $\text{Mod}(F_2)$ -orbit of minimal filling with three curves and unique  $\text{Mod}(F_2)$ -orbit of minimal filling with four curves.

We show that the minimum number of discs in the complement of a *filling pair* of  $F_2$  is two and we construct *filling pair* of  $F_2$  so that the complement is the union of two topological discs. Finally, for given any positive integers  $g, k$  with  $(g, k) \neq (2, 1)$ , we construct filling pair of  $F_g$  such that the complement is the union of  $k$  topological discs.

## 1. INTRODUCTION

Suppose  $F_g$  is the closed surface of genus  $g$  and  $X = \{\gamma_i \mid i = 1, \dots, n\}$  is a nonempty collection of simple closed curves on  $F_g$  such that  $i(\gamma_i, \gamma_j) = |\gamma_i \cap \gamma_j|$  for  $i \neq j$  i.e.,  $\gamma_i$  and  $\gamma_j$  are in minimal position. Here,  $i(\alpha, \beta)$  denotes the geometric intersection number of the closed curves  $\alpha$  and  $\beta$  (see [2]). The set  $X$  is called a filling of the surface  $F_g$  if the complement of  $\bigcup_{i=1}^n \gamma_i$  in  $F_g$  is a disjoint union of topological discs. If the complement is a single disc then we say that  $X$  is a minimal filling of  $F_g$ . Further, if the number of curves in  $X$  is two then we say that  $X$  is a filling pair of  $F_g$ . If  $X$  is a filling of  $F_g$ ,  $g \geq 2$ ,  $T_1(X)$  denotes the number of simple closed curves on  $F_g$  which intersect  $\gamma_1 \cup \dots \cup \gamma_n$  no more than  $n$ . It has been proved in [6] that  $T_1(X) \leq 4g - 2$  and equality holds if and only if  $X$  is minimal filling of  $F_g$ . Filling of surfaces are well studied problem in topology and geometry. It has been studied extensively in [1], [3], [5], [6], [7].

Our motivation to study *fillings* of closed surfaces is the following. The set of all hyperbolic structures on the surface  $F_g$  ( $g \geq 2$ ) upto isometry is called the moduli space of genus  $g$  and is denoted by  $\mathcal{M}_g$ . A systole of a hyperbolic surface is the length of a non-trivial closed geodesic of minimal length. A closed curve or geodesic is called systolic if the systole is realized by its length. It is a well known and difficult problem of constructing a spine of  $\mathcal{M}_g$ , i.e., a deformation retraction in  $\mathcal{M}_g$  of minimal dimension. The subset of  $\mathcal{M}_g$  consisting of all those surfaces whose systolic curves fill the surface is called *Thurston set*, denoted by  $\mathcal{X}_g$ . In [8], Thurston proposed  $\mathcal{X}_g$  as a candidate spine of the moduli space  $\mathcal{M}_g$ . Thurston provided a sketch of a proof that  $\mathcal{X}_g$  is a deformation retract, but it is difficult to

complete the proof. Moreover, the contractibility, connectivity, dimension of the set  $\mathcal{X}_g$  remains open.

Let  $\text{Mod}(F_g)$  denote the mapping class group of the closed surface of genus  $g$ , the group of all orientation preserving homeomorphism upto isotopy (see [2]). It is easy to see that if  $X = \{\gamma_1, \dots, \gamma_n\}$  is a filling of the surface  $F_g$  and  $[f] \in \text{Mod}(F_g)$  the set  $[f] \cdot X = \{f \circ \gamma_1, \dots, f \circ \gamma_n\}$  is also a filling of  $F_g$  with the same number of components in the complement. Let  $I(F_g)$  denote the set of all simple closed curves on  $F_g$ . There is a natural action of  $\text{Mod}(F_g)$  on the quotient space

$$\mathcal{P}_n(F_g) := I(F_g)^n / [(\gamma_1, \dots, \gamma_n) \sim (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)})]$$

where  $\sigma \in \Sigma_n$  is a permutation.

From another point of view one can think the union of the curves in  $X$  as a connected graph on the surface which we denote by  $G_X(F_g)$ . This is in fact a so called fat graph, with all vertices of valence 4. We show:

**Theorem 1.1.** *Suppose  $F_g$  is the closed surface of genus  $g$  and  $X, Y \in \mathcal{P}_n(F_g)$  are two fillings of  $F_g$ . Then they are in the same  $\text{Mod}(F_g)$ -orbit if and only if  $G_X(F_g)$  and  $G_Y(F_g)$  are isomorphic.*

In [6], it has been shown that for all  $g > 2$ , there exist minimal filling pairs on the closed surface  $F_g$  of genus  $g$ . Moreover, there is no minimal filling pair on  $F_2$ . In this situation we have the following question.

**Question.** What is the minimum number of curves in a minimal filling of  $F_2$ ?

The theorem stated below answers the question. Moreover, it says that there exists unique mapping class group orbit of such fillings of  $F_2$ .

**Theorem 1.2.** (1) *There exists a unique  $\text{Mod}(F_2)$ -orbit of triple  $\{\alpha, \beta, \gamma\}$  of curves filling  $F_2$  minimally.*  
 (2) *There exists a unique  $\text{Mod}(F_2)$ -orbit of quadruple  $\{\alpha_i : i = 1, \dots, 4\}$  filling  $F_2$  minimally.*

Now, we study filling pairs on  $F_g$ . Let  $(\alpha, \beta)$  be a filling pair of  $F_g$ . The number of disjoint topological discs in the complement  $F_g \setminus (\alpha \cup \beta)$  is denoted by  $K_g(\alpha, \beta)$ . We define

$$K(F_g) = \min\{K_g(\alpha, \beta) \mid (\alpha, \beta) \text{ is a filling pair of } F_g\}.$$

It has been proved in [6] that  $K(F_g) = 1$  if  $g \geq 3$  and there exists no minimal filling pair of  $F_2$ , i.e.,  $K(F_2) \geq 2$ . In [6], [2], it has been shown that there exists a filling pair  $(\alpha, \beta)$  such that the complement is a union of four pair wise disjoint topological discs which implies  $K(F_2) \leq 4$ . Hence, combining the inequalities we have followed  $2 \leq K(F_2) \leq 4$ . We prove the following theorem which gives the exact value of  $K(F_2)$ .

**Theorem 1.3.** *There exists a filling pair  $(\alpha, \beta)$  of  $F_2$  such that the complement have two components which follows that  $K(F_2) = 2$ .*

Let  $k$  be a positive integer. We have the following question:

**Question.** Does there exist a filling pair  $(\alpha, \beta)$  of  $F_g$ ,  $g \geq 2$  such that  $K_g(\alpha, \beta) = k$ ? We prove:

**Theorem 1.4.** *For every  $k \in \mathbb{N}$  and  $g \geq 2$  with  $(g, k) \neq (2, 1)$ , there exists a filling pair  $(\alpha_k^g, \beta_k^g)$  of  $F_g$  such that the complement  $F_g \setminus (\alpha_k^g \cup \beta_k^g)$  is the disjoint union of  $k$ -topological discs.*

**Organization of the paper:** This paper is organized as follows. In section 2, we give an introduction to fat graphs. We describe the method of construction of the surface associated with a fat graph. Also, we state a lemma which computes the number of boundary components of the surface associated with a given fat graph which is useful in the subsequent sections. In section 3, we study the action of mapping class group on the set of fillings of  $F_g$  and prove Theorem 1.1. In section 4, we focus on the minimal fillings of  $F_2$ . First, we give an independent proof using fat graph to show that there does not exist a minimal filling pairs on  $F_2$ . We conclude this section with a proof of Theorem 1.2. In section 5, we study filling pairs on  $F_g$  for  $g \geq 2$ . We prove Theorem 1.3 and Theorem 1.4 in this section.

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## 2. FAT GRAPHS

A fat graph is a finite graph equipped with a cyclic order of the directed edges going out from each vertex. Before going into the definition of fat graph we recall the definition of graph. The following definition of graph is not the standard one which is used in ordinary graph theory. One can easily see that this definition is equivalent to the standard definition.

**Definition 2.1.** A graph is a triple  $G = (E, \sim, \sigma_1)$  where

- (1)  $E$  is a finite, non-empty set.
- (2)  $\sigma_1 : E \rightarrow E$  is a fixed point free involution.
- (3)  $\sim$  is an equivalence relation on  $E$ .

The set  $E_1 = E/\sigma_1$  of all orbits of  $\sigma_1$  is the set of all un-oriented edges.

*Remark 2.2.* The set  $V = E/\sim$  of all equivalence classes of  $\sim$  is the set of vertices of the graph. For  $v \in V$ , the degree of  $v$  is  $\deg(v) = |v|$ .

**Definition 2.3.** A fat (ribbon) graph is a quadruple  $G = (E, \sim, \sigma_1, \sigma_0)$  where

- (1)  $(E, \sim, \sigma_1)$  is a graph.
- (2)  $\sigma_0$  is a permutation on  $E$  so that each cycle corresponds to an cyclic ordering on the set of oriented edges going out from a vertex.

**2.1. Surface associated to a fat graph.** We construct a topological surface with boundary corresponding to a fat graph  $G$  as described below. We take a closed disc corresponding to each vertex and a rectangle corresponding to each edge. Then we identify the sides of the rectangles with the segments of the boundary of the discs according to the ordering of the edges incident to a vertex. In this way, we obtain the oriented topological surface denoted by  $\Sigma(G)$  corresponding to a given fat graph  $G$ . Thus, we can talk about the number of boundary components, genus and many other topological notions of fat graph. We define  $\sigma_\infty := \sigma_1 * \sigma_0^{-1}$  and denote the set of orbits of  $\sigma_\infty$  by  $E_\infty$ .

**Lemma 2.4.** *Given a fat graph  $G = (E, \sim, \sigma_1, \sigma_0)$ , the number of boundary components of the surface  $\Sigma(G)$  is the number of orbits of  $\sigma_\infty$ .*

We give an explicit example of a fat graph and count its boundary components.

**Example 2.5.** *The graph is given by  $G = (E, \sim, \sigma_1, \sigma_0)$  where*

- (1)  $E = \{\vec{e}_i, \tilde{e}_i \mid i = 1, 2, 3\}$  is the set of oriented edges.

- (2) The involution  $\sigma_1 : E \rightarrow E$  is defined by  $\sigma(\vec{e}_i) = \vec{e}_i$ ,  $i = 1, 2, 3$ .
- (3)  $\sim$  is uniquely determined by the partition  $\{\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}, \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}\}$  of  $E$ .
- (4) The permutation  $\sigma_0 : E \rightarrow E$  is given by  $\sigma_0 = (\vec{e}_1, \vec{e}_3, \vec{e}_2)(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ .

Then we have  $\sigma_\infty = (\vec{e}_1, \vec{e}_2)(\vec{e}_3, \vec{e}_1)(\vec{e}_2, \vec{e}_3)$ . So, it follows from Lemma 2.4 that the number of boundary components of  $G$  is three (see Figure 1).

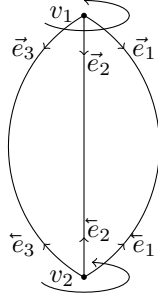


FIGURE 1. The fat graph  $G$ .

**Remark 2.6.** In Example 2.5, if we consider  $\bar{\sigma}_0 = (\vec{e}_1, \vec{e}_2, \vec{e}_3)(\vec{e}_1, \vec{e}_2, \vec{e}_3)$  instead of  $\sigma_0$ , then the number of boundary component is one as  $\sigma_\infty = (\vec{e}_1, \vec{e}_3, \vec{e}_2, \vec{e}_1, \vec{e}_3, \vec{e}_2)$ . Therefore, we conclude that, as ordinary graphs they are the same but as fat graphs they are different.

**Definition 2.7.** A fat graph is called decorated if the degree of each vertex is even and at least 4.

**Definition 2.8.** Let  $G = (E, \sim, \sigma_1, \sigma_0)$  and  $G' = (E', \sim', \sigma'_1, \sigma'_0)$  be two fat graphs.  $G$  and  $G'$  are said to be isomorphic if there exists a bijective function  $f : E \rightarrow E'$  such that:

- (1) For  $x_1, x_2 \in E$ ,  $x_1 \sim x_2$  if and only if  $f(x_1) \sim' f(x_2)$ .
- (2) The following diagram commutes:

$$\begin{array}{ccc}
 E & \xrightarrow{f} & E' \\
 \sigma_1 \downarrow & \curvearrowright & \downarrow \sigma'_1 \\
 E & \xrightarrow{f} & E'
 \end{array}$$

- (3)  $(x_1, x_2, \dots, x_n)$  is the cyclic ordering on the set of edges going out from  $v = \{x_1, x_2, \dots, x_n\}$  if and only if  $(f(x_1), f(x_2), \dots, f(x_n))$  is the cyclic ordering on the set of edges going out from  $v' = \{f(x_1), f(x_2), \dots, f(x_n)\}$ .

### 3. MAPPING CLASS GROUP ORBITS OF FILLINGS

Let  $I(F_g)$  denote the set of all isotopy classes of simple closed curves on  $F_g$ . For  $n \in \mathbb{N}$ ,  $I(F_g)^n$  denotes the cartesian product of  $n$  copies of  $I(F_g)$ . We define a relation  $\sim$  on  $I(F_g)^n$  by  $(\gamma_1, \gamma_2, \dots, \gamma_n) \sim (\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \dots, \gamma_{\sigma(n)})$  for some  $\sigma \in \Sigma_n$ . The mapping class group  $\text{Mod}(F_g)$  acts on  $I(F_g)^n$  by  $[f] \cdot (\gamma_1, \gamma_2, \dots, \gamma_n) = (f \circ \gamma_1, \dots, f \circ \gamma_n)$ . This action descends to an action on the quotient space  $\mathcal{P}_n(F_g) :=$

$I(F_g)^n/\sim$ . Let  $X = (\alpha_1, \dots, \alpha_n) \in \mathcal{P}_n(F_g)$  be a filling of  $F_g$  and  $[f] \in \text{Mod}(F_g)$ , then  $[f] \cdot X = (f \circ \alpha_1, \dots, f \circ \alpha_n)$  is also a filling of  $F_g$  with the same number of topological discs in the complement.

Suppose  $X = \{\alpha_1, \dots, \alpha_n\}$  is a filling of  $F_g$ . Then we can think of the union of the curves in  $X$  as a 4-regular decorated fat graph denoted by  $G_X(F_g)$  which is described below.

- (1) The intersection points  $\alpha_i \cap \alpha_j, i \neq j \in \{1, \dots, n\}$  are the vertices.
- (2) The sub-arcs of  $\alpha_i$ 's joining the vertices are the edges.
- (3) The cyclic order on the set of edges incident at each vertex is uniquely determined by the orientation of the surface.

Conversely, suppose  $G$  is a given decorated 4-regular fat graph with standard cycles  $C_i; i = 1, \dots, n$ . We obtain the closed surface  $F(G)$  by capping each boundary component of  $G$  by a topological disc. Then  $X_G = \{C_i | i = 1, \dots, n\}$  is a filling of  $F(G)$ .

**Theorem 3.1.** *Suppose  $X, Y \in \mathcal{P}_n(F_g)$  are two fillings of  $F_g$ . Then  $X, Y$  are in the same  $\text{Mod}(F_g)$ -orbit if and only if  $G_X(F_g)$  and  $G_Y(F_g)$  are isomorphic as fat graphs.*

*Proof.* Suppose  $X$  and  $Y$  are in the same  $\text{Mod}(F_g)$ -orbit which follows that there exists  $[f] \in \text{Mod}(F_g)$  such that  $Y = f \cdot X$ . The restriction  $\tilde{f} = f|_{G_X(F_g)}$  of the homeomorphism  $f : F_g \rightarrow F_g$  gives a fat graph isomorphism

$$\tilde{f} : G_X(F_g) \rightarrow G_Y(F_g).$$

Conversely, if  $\tilde{f} : G_X(F_g) \rightarrow G_Y(F_g)$  is an isomorphism then the isomorphism can be extended to a homeomorphism  $f : F_g \rightarrow F_g$  such that  $Y = f \cdot X$  which implies that  $X$  and  $Y$  are in the same  $\text{Mod}(F_g)$ -orbit.  $\square$

#### 4. FILLINGS OF $F_2$

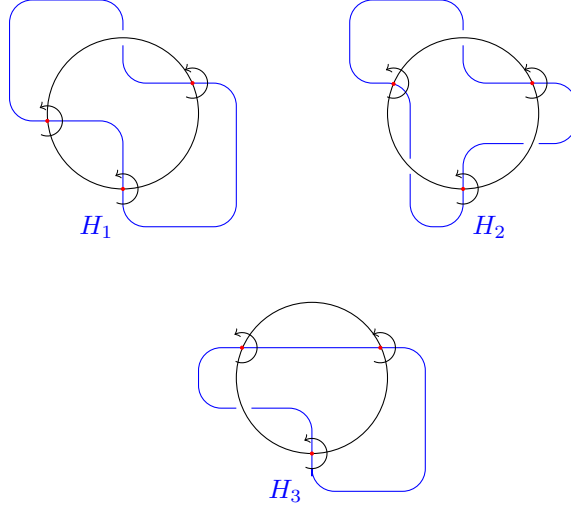
**Lemma 4.1.** *Let  $G$  be a 4-valent decorated fat graph with three vertices and two standard cycles. Then the number of boundary components in  $G$  is at least two.*

*Proof.* Let  $C_i, i = 1, 2$  be the standard cycles of  $G$ . Then each  $C_i$  is simple and consists of three edges. There are three such 4-valent fat graphs up to isomorphism with three vertices and two standard cycles which are denoted by  $H_i, i = 1, 2, 3$  (see Figure 2). For each  $i \in \{1, 2, 3\}$  the graph  $H_i$  has three boundary components.  $\square$

**Corollary 4.2.** *There exists no minimal filling pair of  $F_2$ .*

*Proof.* Suppose there is a minimal filling pair  $(\alpha, \beta)$  of  $F_2$ . Then, we define  $G := \alpha \cup \beta$ . We regard  $G$  as a decorated fat graph where the intersection points of  $\alpha, \beta$  are the vertices, the sub-segments of  $\alpha$  and  $\beta$  joining two vertices are the edges. The cyclic order on the set of edges incident at each vertex is determined by the orientation of the surface.

In another way, we can think  $G$  as the 1-skeleton of a cellular decomposition of  $F_2$ . In the cell decomposition the number of 0-cells is  $i(\alpha, \beta)$ . The valency condition follows that the number of 1-cells is  $2i(\alpha, \beta)$  and from the minimality condition we have the number of 2-cells is 1. Therefore, the Euler characteristic argument implies that  $i(\alpha, \beta) = 3$ . Hence,  $G$  is a 4-regular decorated fat graph

FIGURE 2. The graphs  $H_i, i = 1, 2, 3$ .

with three vertices, two standard cycles and a single boundary component which contradicts Lemma 4.1.  $\square$

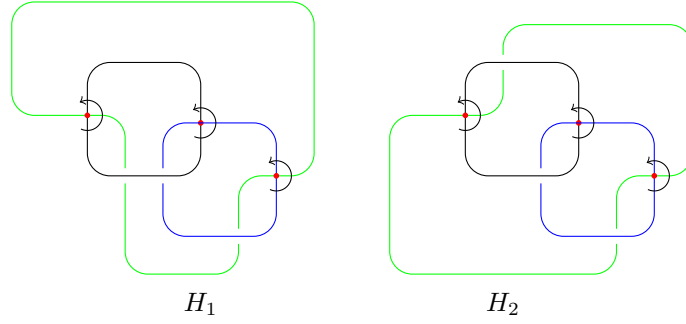
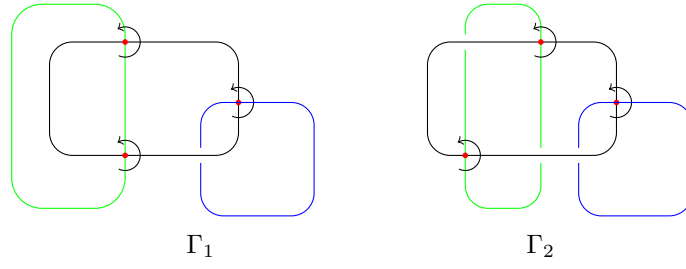
**Theorem 4.3.** *There exists a unique  $\text{Mod}(F_2)$ -orbit of minimal filling triple  $\{\alpha, \beta, \gamma\}$  and quadruple  $\{\alpha_i \mid i = 1, 2, 3, 4\}$  of  $F_2$ .*

*Proof. Filling triple:* Suppose  $\{\alpha, \beta, \gamma\}$  is a minimal filling triple of  $F_2$ . We define  $G := \alpha \cup \beta \cup \gamma$ . Then  $G$  is a 4-regular decorated fat graph on  $F_2$  with three standard cycles  $\alpha, \beta$  and  $\gamma$ . In another point of view  $G$  is the 1-skeleton of a cellular decomposition of  $F_2$ . The minimality of the filling, Euler characteristic argument and regularity of the graph  $G$  imply that the number of vertices is 3 and the number of edges is 6. Therefore, to prove the theorem it suffices to prove that there exists a unique 4-regular decorated fat graph  $G$  with three vertices, three standard cycles and single boundary component.

Let  $C_i, i = 1, 2, 3$  be the standard cycles of  $G$ . There are two cases to be considered. Case 1. In this case, we consider that each standard cycle  $C_i$  consists of two edges. Up to isomorphism there are only two distinct such fat graphs  $H_1, H_2$  which are given in Figure 3. For each  $i = 1, 2$ , the fat graph  $H_i$  has three boundary components. Therefore, this case is not possible.

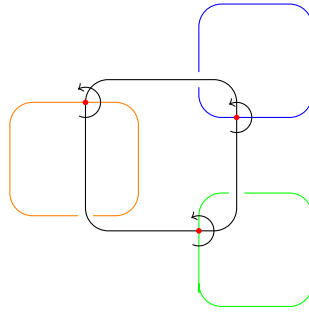
Case 2. In this case assume that there is a cycle  $C_i$  which consists of three edges. Without loss of generality assume that  $C_1$  consists of three edges. The Euler characteristic argument implies that other two standard cycles  $C_2$  and  $C_3$  consist of two edges and one edge respectively. The similar argument as in the case 1 it follows that up to isomorphism there are two distinct such fat graphs  $\Gamma_1, \Gamma_2$  which are given in Figure 4. The graph  $\Gamma_1$  has three boundary components and  $\Gamma_2$  has one boundary component. Therefore, by construction  $G = \Gamma_2$  is the unique fat graph which satisfies the theorem.

**Filling quadruple:** To prove the second part, it suffices to show that there exist unique 4-valent decorated fat graph  $G$  with three vertices and four standard cycles


 FIGURE 3. The graphs  $H_i, i = 1, 2$ .

 FIGURE 4. The graphs  $\Gamma_i, i = 1, 2$ .

and single boundary component. Suppose  $C_i, i = 1, 2, 3, 4$  are the standard cycles of  $G$ . As in the proof of the first part there are two cases to be considered.

Case 1. Suppose there is a standard cycle consists of three edges. So, without loss of generality assume that  $C_1$  has length 3. The number of edges in  $G$  is six. Therefore, it follows that for each  $i = 2, 3, 4, C_i$  is a loop. There are three vertices on  $C_1$  and  $C_i, i = 2, 3, 4$  are the loops at the vertices on  $C_1$ . Such a graph  $H$  is uniquely determined (up to isomorphism) and is given in the Figure 5. The number


 FIGURE 5. The graph  $H$ .

of boundary components in  $H$  (Figure 5) is three. So this case is not possible.

Case 2. In this case, we assume that there are no standard cycle of length three. The only possibility is the following: there are two cycles of length two and two cycles of length one. Such a 4-valent graph  $K$  is uniquely (up to isomorphism) determined and is given in the Figure 6 which have a single boundary component. Hence  $G = K$  is the unique 4-valent decorated fat graph satisfies the theorem.

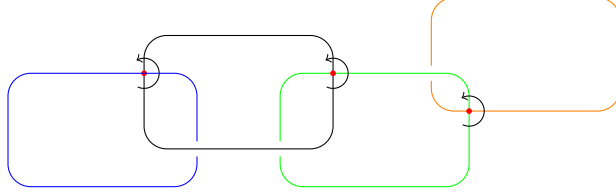


FIGURE 6. The graph  $K$ .

□

*Remark 4.4.* For  $n \geq 5$  there does not exist a connected 4-valent decorated fat graph with  $n$  standard cycles and three vertices which follows that there does not exist any minimal filling  $X = \{\gamma_i \mid i = 1, 2, \dots, n\}$  of  $F_2$  if  $n \geq 5$ .

**Theorem 4.5.** *There exists a filling pair  $(\alpha, \beta)$  of  $F_2$  such that the complement is union of two topological discs which follows that  $K(F_2) = 2$ .*

*Proof.* Let  $(\alpha, \beta)$  be a filling pair of  $F_2$  such that the complement is disjoint union of two topological discs. Consider the graph  $G = \alpha \cup \beta$  on  $F_2$ . It follows from the Euler characteristic argument that the number of vertices in  $G$  is four and the number of edges is eight. The standard cycles of  $G$  are the cycles corresponding to the closed curves  $\alpha$  and  $\beta$ . The number of boundary components of  $G$  is the same as the number of components in the complement of  $\alpha \cup \beta$  in  $F_2$  which is equal to two.

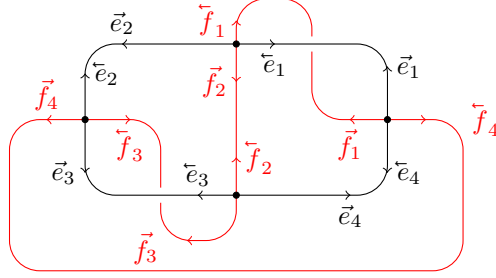
The theorem boils down once we have shown the existence of a 4-regular graph  $G$  as above. Note that, if we have such a graph, we cap the boundary components by topological discs and obtain  $F_2$ . Consider the graph  $G = (E, \sim, \sigma_1, \sigma_0)$  described below (see Figure 7).

- (1)  $E = \{\vec{e}_i, \vec{e}_i, \vec{f}_i, \vec{f}_i \mid i = 1, \dots, 4\}$ .
- (2)  $\sim$  is uniquely determined by the partition  $V = \{v_i \mid i = 1, \dots, 4\}$  of  $E$  where  $v_1 = \{\vec{e}_1, \vec{f}_1, \vec{e}_4, \vec{f}_4\}$ ,  $v_2 = \{\vec{e}_2, \vec{f}_2, \vec{e}_1, \vec{f}_1\}$ ,  $v_3 = \{\vec{e}_3, \vec{f}_3, \vec{e}_2, \vec{f}_4\}$ ,  $v_4 = \{\vec{e}_4, \vec{f}_2, \vec{e}_3, \vec{f}_3\}$ .
- (3)  $\sigma_1(\vec{e}_i) = \vec{e}_i$ ,  $\sigma_1(\vec{f}_i) = \vec{f}_i$ ,  $i = 1, \dots, 4$ .
- (4)  $\sigma_0 = (\vec{e}_1, \vec{f}_1, \vec{e}_4, \vec{f}_4)(\vec{e}_2, \vec{f}_2, \vec{e}_1, \vec{f}_1)(\vec{e}_3, \vec{f}_3, \vec{e}_2, \vec{f}_4)(\vec{e}_4, \vec{f}_2, \vec{e}_3, \vec{f}_3)$ .

We have  $\sigma_\infty = (\vec{e}_1, \vec{f}_2, \vec{e}_4, \vec{f}_1, \vec{e}_1, \vec{f}_4, \vec{e}_2, \vec{f}_1)(\vec{e}_2, \vec{f}_3, \vec{e}_3, \vec{f}_4, \vec{e}_4, \vec{f}_3, \vec{e}_3, \vec{f}_2)$  which follows that the number of boundary components in the fat graph  $G$  is two (see Lemma 4.1).

□



FIGURE 7. The graph  $G$ .5. FILLING PAIRS OF  $F_g$ 

**Theorem 5.1.** *For every  $k \in \mathbb{N}$  and  $g \geq 2$  with  $(g, k) \neq (2, 1)$ , there exists a filling pair  $(\alpha_k^g, \beta_k^g)$  of  $F_g$  such that the complement of  $\alpha_k^g \cup \beta_k^g$  in  $F_g$  is the disjoint union of  $k$ -topological discs.*

Before going to the proof of Theorem 5.1, we prove two lemmas which are essential for the proof of the theorem. The lemmas are exactly the particular cases of Theorem 5.1 when  $g = 2$  and  $g = 3$ .

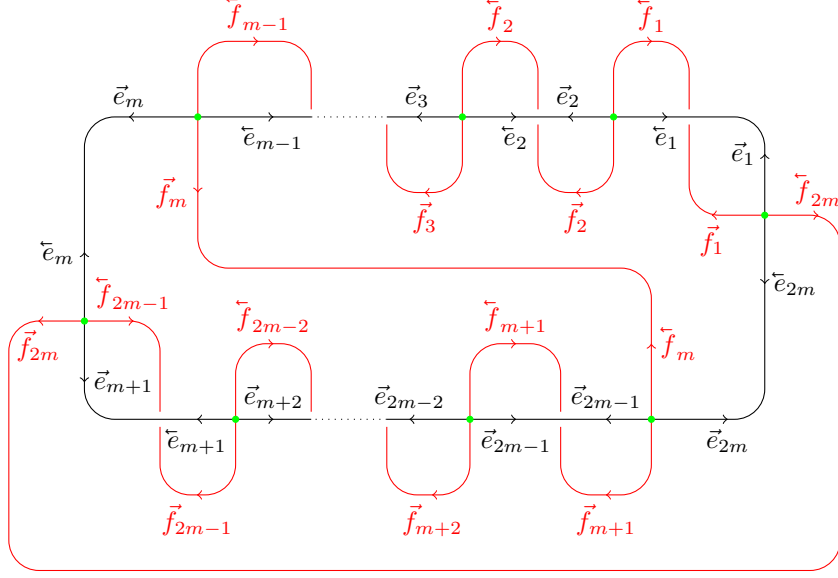
**Lemma 5.2.** *For every  $k(\geq 2) \in \mathbb{Z}$ , there exists a filling pair  $(\alpha_k^2, \beta_k^2)$  of  $F_2$  such that the complement of  $(\alpha_k^2 \cup \beta_k^2)$  in  $F_2$  is the union of  $k$ -topological discs.*

*Proof.* Suppose that there exists a filling pair  $(\alpha_k^2, \beta_k^2)$  of  $F_2$  which satisfies the lemma. Now, consider the graph  $G_k^2 := \alpha_k^2 \cup \beta_k^2$  on  $F_2$ . It follows from the Euler characteristic argument that  $G_k^2$  has  $k + 2$  vertices and  $k$  boundary components. The number of standard cycles in  $G_k^2$  is two which are correspond to  $\alpha_k^2$  and  $\beta_k^2$ . Conversely, if we have a fat graph as above then by attaching topological discs along the boundary components we obtain closed surface  $F_2$  of genus 2 and the standard cycles provide us a filling pair for the lemma. Therefore, to prove the lemma, it suffices to construct a graph  $G_k^2 = (E, \sim, \sigma_1, \sigma_0)$  as above for each  $k \geq 2$ . We consider the two cases.

Case 1. Let  $k$  be an even integer and  $k = 2n$  for some  $n \in \mathbb{N}$  (see Figure 8). Then the number of vertices is  $2m$  where  $m = n + 1$ . Now, the graph  $G_k^2$  is described below.

- (1)  $E = \{\vec{e}_i, \vec{e}_i, \vec{f}_i, \vec{f}_i | i = 1, \dots, 2m\}$ .
- (2)  $\sim$  is determined by the partition  $V = \{v_i | 1 \leq i \leq 2m\}$  where  $v_1 = \{\vec{e}_1, \vec{f}_1, \vec{e}_{2m}, \vec{f}_{2m}\}$ ,  $v_i = \{\vec{e}_i, \vec{f}_i, \vec{e}_{i-1}, \vec{f}_{i-1}\}$ ;  $2 \leq i \leq m$ , and  $v_i = \{\vec{e}_i, \vec{f}_{3m-i}, \vec{e}_{i-1}, \vec{f}_{3m-i+1}\}$ ;  $m+1 \leq i \leq 2m$ .
- (3)  $\sigma_1(\vec{e}_i) = \vec{e}_i$  and  $\sigma_1(\vec{f}_i) = \vec{f}_i$ ,  $i = 1, 2, \dots, 2m$ .
- (4)  $\sigma_0 = C_1 \cdots C_{2m}$  where  $C_1 = (\vec{e}_1, \vec{f}_1, \vec{e}_{2m}, \vec{f}_{2m})$ ,  $C_i = (\vec{e}_i, \vec{f}_i, \vec{e}_{i-1}, \vec{f}_{i-1})$  for  $i = 2, \dots, m$  and  $C_i = (\vec{e}_i, \vec{f}_{3m-i}, \vec{e}_{i-1}, \vec{f}_{3m-i+1})$  for  $i = m+1, \dots, 2m$ .

Now, we count the boundary components of  $G_k^2$ , equivalently orbits of  $\sigma_\infty$ . The orbits of  $\sigma_\infty$  are  $\partial_i = (\vec{e}_i, \vec{f}_{i+1}, \vec{e}_{i+1}, \vec{f}_i)$  for  $i = 1, \dots, m-2$ ,  $\partial_{m-1} = (\vec{e}_{m-1}, \vec{f}_m, \vec{e}_{2m}, \vec{f}_1, \vec{e}_1, \vec{f}_{2m}, \vec{e}_m, \vec{f}_{m-1})$ ,  $\partial_m = (\vec{e}_m, \vec{f}_{2m-1}, \vec{e}_{m+1}, \vec{f}_{2m}, \vec{e}_{2m}, \vec{f}_{m+1}, \vec{e}_{2m-1}, \vec{f}_m)$ , and  $\partial_j = (\vec{e}_j, \vec{f}_{3m-j-1}, \vec{e}_{j+1}, \vec{f}_{3m-j})$ ;  $j = m+1, \dots, 2m-2$ . So, there are  $k$  orbits of  $\sigma_\infty$ .

FIGURE 8. The graph  $G_k^2$  ( $k$  is even).

Case 2. Here we consider  $k = 2n + 1, n \in \mathbb{N}$ . The graph  $G_k^2 = (E, \sim, \sigma_1, \sigma_0)$  is described below (see Figure 9).

- (1)  $E = \{\vec{e}_i, \vec{e}_i, \vec{f}_i, \vec{f}_i | i = 1, \dots, 2n + 3\}$ .
- (2)  $\sim$  is uniquely determined by the partition  $V = \{v_i | i = 1, \dots, 2n + 3\}$  of  $E$  where  $v_1 = \{\vec{e}_1, \vec{f}_1, \vec{e}_{2n+3}, \vec{f}_{2n+3}\}$ ,  $v_i = \{\vec{e}_i, \vec{f}_{2n+3-i}, \vec{e}_{i-1}, \vec{f}_{2n+4-i}\}$  for  $i = 2, \dots, 2n + 2$  and  $v_{2n+3} = \{\vec{e}_{2n+3}, \vec{f}_{2n+3}, \vec{e}_{2n+2}, \vec{f}_{2n+2}\}$ .
- (3)  $\sigma_1(\vec{e}_i) = \vec{e}_i$ , and  $\sigma_1(\vec{f}_i) = \vec{f}_i$  for  $i = 1, \dots, 2n + 3$ .
- (4)  $\sigma_0 = C_1 C_2 \dots C_{2n+3}$  where  $C_1 = (\vec{e}_1, \vec{f}_1, \vec{e}_{2n+3}, \vec{f}_{2n+3})$ ,  $C_i = (\vec{e}_i, \vec{f}_{2n+3-i}, \vec{e}_{i-1}, \vec{f}_{2n+4-i})$  for  $i = 2, \dots, 2n + 2$  and  $C_{2n+3} = (\vec{e}_{2n+3}, \vec{f}_{2n+3}, \vec{e}_{2n+2}, \vec{f}_{2n+2})$ .

The orbit of  $\sigma_\infty$  are  $D_0 = \{\vec{e}_1, \vec{f}_{2n+1}, \vec{e}_2, \vec{f}_{2n+2}, \vec{e}_{2n+2}, \vec{f}_2, \vec{e}_{2n-1}, \vec{f}_1\}$ ,  $D_1 = \{\vec{f}_1, \vec{e}_{2n+2}, \vec{f}_{2n+3}, \vec{e}_{2n+3}, \vec{f}_{2n+2}, \vec{e}_1, \vec{f}_{2n+3}, \vec{e}_{2n+3}\}$ ,  $D_i = \{\vec{e}_i, \vec{f}_{2k+2-i}, \vec{e}_{i+1}, \vec{f}_{2k+3-i}\}$  for  $i = 2, \dots, 2k$ . Hence, there are  $k$  orbits of  $\sigma_\infty$  which is equal to the number of boundary components of the fat graph  $G_k^2$ .  $\square$

**Lemma 5.3.** For every  $k(\geq 1) \in \mathbb{Z}$ , there exists a filling pair  $(\alpha_k^3, \beta_k^3)$  of  $F_3$  such that the complement  $F_3 \setminus (\alpha_k^3 \cup \beta_k^3)$  is the disjoint union of  $k$  topological discs.

*Proof.* The proof of this lemma is similar as the proof of Lemma 5.2. We construct 4-regular fat graph  $G_k^3 = (E, \sim, \sigma_1, \sigma_0)$  with  $k + 4$  vertices, two standard cycles and  $k$  boundary components. As before, we consider two cases.

Case 1. In this case we consider  $k$  is an odd integer. Let  $k = 2m - 1$  for some  $m \in \mathbb{N}$ . The graph is described below.

- (1)  $E = \{\vec{e}_i, \vec{f}_i, \vec{e}_i, \vec{f}_i | i = 1, \dots, 2m + 3\}$ .
- (2)  $\sim$  is uniquely determined by the partition  $V = \{v_i | i = 1, \dots, 2m + 3\}$  of  $E$  where  $v_1 = \{\vec{e}_1, \vec{f}_1, \vec{e}_{2m+3}, \vec{f}_{2m+3}\}$ ,  $v_i = \{\vec{e}_i, \vec{f}_i, \vec{e}_{i-1}, \vec{f}_{i+1}\}$  for  $i =$



**Example 5.4.** The 4-regular fat graph  $G$  given in Figure 10 has five nodes, one boundary component and two standard cycles. Therefore, the pair of simple closed curves  $(\alpha, \beta)$  in  $\Sigma(G)$  which are associated to the standard cycles on  $G$  is a minimal filling pair. It follows from the Euler characteristic argument that  $\Sigma(G)$  is homeomorphic to  $F_3$ .

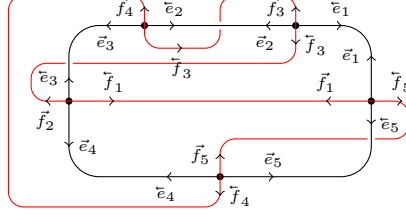


FIGURE 10. The graph  $G$ .

**Example 5.5.** In this example we Consider a fat graph  $\Gamma = (E, \sim, \sigma_1, \sigma_0)$  given below and compute its boundary components.

- (1)  $E = \{\vec{y}_i, \vec{x}_i, \vec{y}_i, \vec{x}_i \mid i = 1, \dots, 6\}$ .
- (2)  $\sim$  is defined by the partition  $U = \{u_i \mid i = 1, \dots, 6\}$  of  $E$  where  $u_1 = \{\vec{y}_1, \vec{x}_6, \vec{y}_6, \vec{x}_1\}$ ,  $u_2 = \{\vec{y}_2, \vec{x}_3, \vec{y}_1, \vec{x}_2\}$ ,  $u_3 = \{\vec{y}_3, \vec{x}_2, \vec{y}_2, \vec{x}_1\}$ ,  $u_4 = \{\vec{y}_4, \vec{x}_3, \vec{y}_3, \vec{x}_4\}$ ,  $u_5 = \{\vec{y}_5, \vec{x}_6, \vec{y}_4, \vec{x}_5\}$ ,  $u_6 = \{\vec{y}_6, \vec{x}_5, \vec{y}_5, \vec{x}_4\}$ .
- (3)  $\sigma_1(\vec{x}_i) = \vec{x}_i, \sigma_1(\vec{y}_i) = \vec{y}_i, i = 1, \dots, 6$ .
- (4)  $\sigma_0 = (\vec{y}_1, \vec{x}_6, \vec{y}_6, \vec{x}_1)(\vec{y}_2, \vec{x}_3, \vec{y}_1, \vec{x}_2)(\vec{y}_3, \vec{x}_2, \vec{y}_2, \vec{x}_1)(\vec{y}_4, \vec{x}_3, \vec{y}_3, \vec{x}_4)(\vec{y}_5, \vec{x}_6, \vec{y}_4, \vec{x}_5)(\vec{y}_6, \vec{x}_5, \vec{y}_5, \vec{x}_4)$ .

The boundary components of  $\Gamma$  are  $(\vec{y}_1, \vec{x}_3, \vec{y}_4, \vec{x}_6)$ ,  $(\vec{y}_2, \vec{x}_2, \vec{y}_1, \vec{x}_1, \vec{y}_2, \vec{x}_2, \vec{y}_3, \vec{x}_3)$ ,  $(\vec{y}_5, \vec{x}_5, \vec{y}_4, \vec{x}_4, \vec{y}_5, \vec{x}_5, \vec{y}_6, \vec{x}_6)$  and  $(\vec{y}_3, \vec{x}_1, \vec{y}_6, \vec{x}_4)$ .

Let  $G_k^g$  denote a 4-regular fat graph with two standard cycles,  $k$  boundary components and such that if we glue discs along the boundary components the resulting surface is  $F_g$ . Therefore, the standard cycles of the graph provide us a filling pair of  $F_g$  such that the complement is the disjoint union of  $k$  topological discs. It follows from the Euler characteristic number argument that the number of vertices in  $G_k^g$  is  $m = 2g - 2 + k$ .

*Proof of Theorem 5.1.* We prove the theorem by mathematical induction. It is already proved in Lemma 5.2 and Lemma 5.3 for the case when  $g = 2$  and  $g = 3$  respectively. Suppose  $G_k^g = (E, \sim, \sigma_1, \sigma_0)$  is given, we attach the graph  $\Gamma$  at the vertex  $u_1$  (see Example 5.5) with  $G_k^g$  at the vertex  $v_1$  and obtain  $G_k^g \#_{(v_1, u_1)} \Gamma$ . Let  $E = \{\vec{e}_i, \vec{e}_i, \vec{f}_i, \vec{f}_i \mid i = 1, \dots, 2g - 2 + k\}$ . We label the graph such that  $v_1 = \{\vec{e}_1, \vec{f}_1, \vec{e}_m, \vec{f}_m\}$  is a vertex. The graph  $G_k^g \#_{(v_1, u_1)} \Gamma = (E', \sim', \sigma'_1, \sigma'_0)$  is described below.

- (1)  $E' = \{\vec{e}'_i, \vec{e}'_i, \vec{f}'_i, \vec{f}'_i \mid i = 1, 2, \dots, m + 4\}$  where  $\vec{e}'_1 = \vec{x}_1 * \vec{e}_1$ ,  $\vec{e}'_i = \vec{e}_i$  for  $i = 2, \dots, m - 1$ ,  $\vec{e}'_m = \vec{e}_m * \vec{x}_6$ ,  $\vec{e}'_{m+1} = \vec{x}_5$ ,  $\vec{e}'_{m+2} = \vec{x}_4$ ,  $\vec{e}'_{m+3} = \vec{x}_3$ ,  $\vec{e}'_{m+4} = \vec{x}_2$  and  $\vec{f}'_1 = \vec{y}_6 * \vec{f}_1$ ,  $\vec{f}'_i = \vec{f}_i$  for  $i = 2, \dots, m - 1$ ,  $\vec{f}'_m = \vec{f}_m * \vec{y}_1$ ,  $\vec{f}'_{m+1} = \vec{y}_2$ ,  $\vec{f}'_{m+2} = \vec{y}_3$ ,  $\vec{f}'_{m+3} = \vec{y}_4$ ,  $\vec{f}'_{m+4} = \vec{y}_5$ .

- (2) Let for  $v = (a_1, a_2, a_3, a_4)$  be a vertex in  $V \cup U \setminus \{v_1, u_1\}$  then we define  $\tilde{v} = (a'_1, a'_2, a'_3, a'_4)$  where  $a'_p = a_p$  if  $a_p \in \{\vec{e}_i, \vec{e}_i, \vec{f}_i, \vec{f}_i, \vec{x}_j, \vec{x}_j, \vec{y}_j, \vec{y}_j | i = 2, \dots, m; j = 2, \dots, 5\}$  and  $a'_p = \tilde{x}_1 * \vec{e}_1$  if  $a_p = \tilde{x}_1$  or  $\vec{e}_1$  and so on for  $p = 1, \dots, 4$ . The set of vertices of  $G_k^{g+2}$  is given by

$$V' = \{\tilde{v} | v \in V \cup U \setminus \{v_1, u_1\}\}.$$

- (3)  $\sigma'_1(\vec{e}'_i) = \vec{e}'_i$  and  $\sigma'_1(\vec{f}'_i) = \vec{f}'_i$  where  $i = 1, 2, \dots, m+4$ .  
 (4) The permutation  $\sigma'_0$  is the product of pairwise disjoint cycles given below

$$\sigma'_0 = \prod_{\tilde{v} \in V'} \tilde{v}.$$

Let  $\partial$  be a boundary component of  $G_k^g$  and  $\partial = (a_1, a_2, \dots, a_l)$  which is a finite sequence of directed edges written uniquely up to cyclic order. We define  $\tilde{\partial}$  by following.

**Case 1.** If,  $a_j \in \{\vec{e}_i, \vec{e}_i, \vec{f}_i, \vec{f}_i | i = 2, \dots, m-1\}$  for all  $j = 1, \dots, l$  then we define  $\tilde{\partial} = \partial$ .

**Case 2.** If  $\vec{e}_1$  is in  $\partial$  then  $\vec{f}_1$  must be in  $\partial$  which is counted immediate before  $\vec{e}_1$ . We replace the subsequence  $(\vec{f}_1, \vec{e}_1)$  in  $\partial$  by the sequence  $(\vec{f}_1 * \vec{y}_6, \vec{x}_4, \vec{y}_3, \vec{x}_1 * \vec{e}_1)$ . Similarly, if  $\partial$  contains  $(\vec{e}_m, \vec{f}_1)$ ,  $(\vec{f}_m, \vec{f}_m)$ ,  $(\vec{e}_1, \vec{f}_m)$ , then we replace them by  $(\vec{e}_m * \vec{x}_6, \vec{y}_5, \vec{x}_5, \vec{y}_4, \vec{x}_4, \vec{y}_5, \vec{x}_5, \vec{y}_6 * \vec{f}_1)$ ,  $(\vec{f}_m * \vec{y}_1, \vec{x}_3, \vec{y}_4, \vec{x}_6 * \vec{e}_m)$ ,  $(\vec{e}_1 * \vec{x}_1, \vec{y}_2, \vec{x}_2, \vec{y}_3, \vec{x}_3, \vec{y}_2, \vec{x}_2, \vec{y}_1 * \vec{f}_m)$  respectively and the obtained new finite sequence of directed edges is  $\tilde{\partial}$ .

Suppose  $\{\partial_i | i = 1, \dots, k\}$  is the set of boundary components of  $G_k^g$ . Then  $\{\tilde{\partial}_i | i = 1, \dots, k\}$  is the set of all boundary components of  $G_k^g \#_{(v_1, u_1)} \Gamma$ . Therefore, the number of boundary components in  $G_k^g \#_{(v_1, u_1)} \Gamma$  is  $k$ . The standard cycles of  $G_k^g \#_{(v_1, u_1)} \Gamma$  are  $e'_1 * e'_2 * \dots * e'_{m+4}$  and  $f'_1 * f'_2 * \dots * f'_{m+4}$ . We define

$$G_k^{g+2} := G_k^g \#_{(v_1, u_1)} \Gamma.$$

□

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